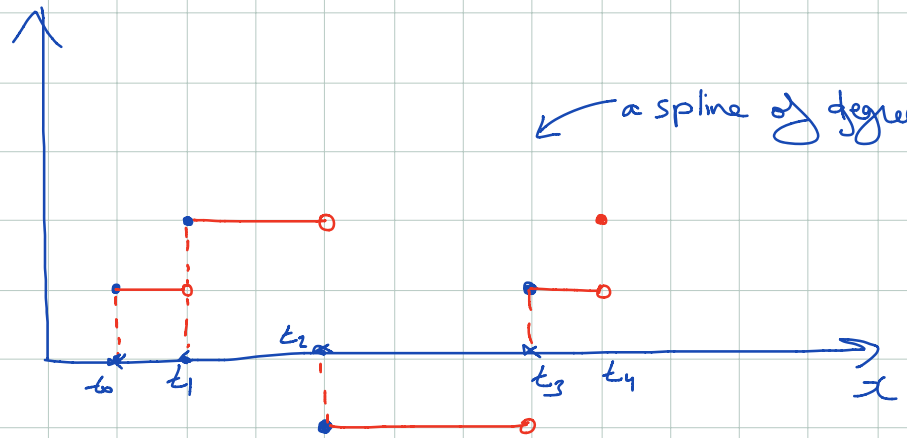
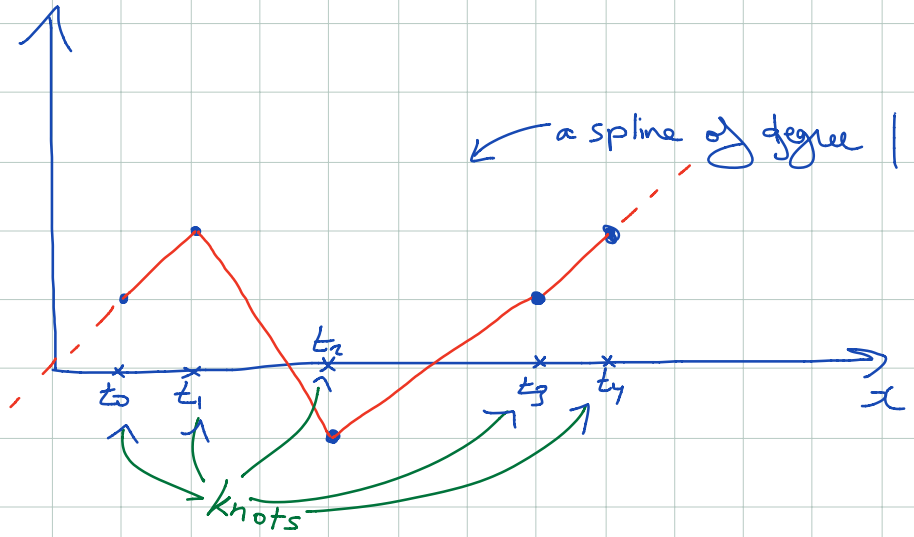
Section 6.4 : Spline Interpolation





A spline function consists of polynomial pieces on sub intervals. They are joined together in a way that satisfies continuity conditions.

- on $[t_{i-1}, t_i)$, S is a polynomial of degree $\leq k$
- S has a continuous $(k-1)^{st}$ derivative on $[t_0, t_n]$

e.g. Spline of degree 0 : piecewise constant
Spline of degree 1 : piecewise linear

Cubic Splines (Splines with degree $k=3$)

Denote by $S(x)$ the cubic spline interpolating some data (t_i, y_i), $0 \le i \le n$

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

so $S_{i-1}(t_i) = S_i(t_i) \ (= y_i)$ for $i = 1, \dots, n-1$

we also ask that S' & S'' are continuous

Is this possible?

- we have 2-equations per S_i : $S_i(t_i) = y_i$ & $S_i(t_{i+1}) = y_{i+1}$

$\Rightarrow 2n$ equations

- we also have $S_i'(t_i) = S_{i-1}'(t_i)$ for $i = 1, \dots, n-1$

$\Rightarrow 2(n-1)$ equations $S_i''(t_i) = S_{i-1}''(t_i)$ for $i = 1, \dots, n-1$

$\Rightarrow$ Total of $4n-2$ equations

But each $S_i(x)$ has 4-degrees of freedom $(a+bx+cx^2+dx^3)$

$\Rightarrow$ $4n$ - unknowns

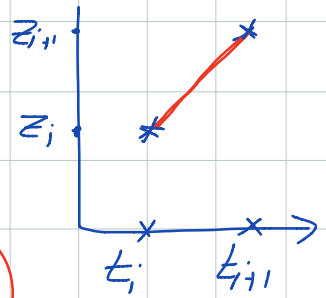
We're left with 2 degrees of freedom.

Ok, let's now figure out the S_i's.

- Let $z_i = S_i''(t_i)$ $\quad \left(z_i = \lim_{x \to t_i^+} S''(x) = \lim_{x \to t_i^-} \{S''(x)\} \right.$
$\left. i = 1, \dots, n-1 \right)$

moreone $S_i''(x)$ is a linear function with $S_i''(t_i) = z_i$, $S_i''(t_{i+1}) = z_{i+1}$

So



$$S_i''(x) = \underbrace{\frac{t_{i+1} - x}{t_{i+1} - t_i}}_{h_i} \cdot z_i + \frac{x - t_i}{t_{i+1} - t_i} \cdot z_{i+1}$$

$$\Rightarrow S_i(x) = \frac{z_i}{6h_i}(t_{i+1}-x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + C(x-t_i) + D(t_i-x)$$

↑ integrate twice

constants of integration

To find C & D: $S_i(t_i) = y_i$, $S_i(t_{i+1}) = y_{i+1}$

$$\Rightarrow C = \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right) \quad \& \quad D = \frac{y_i}{h_i} - \frac{z_i h_i}{6}$$

- Now, to find the z_i's note that $S'_{i-1}(t_i) = S'_i(t_i)$

Differentiate the expressions for $S_i(x)$, $S_{i-1}(x)$ and plug in t_i :

$$\Rightarrow h_{i-1}z_{i-1} + 2(h_i + h_{i-1})z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1}-y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

for $i = 1, \dots, n-1$

$\Rightarrow n+1$ unknowns $(z_0 \dots z_n)$
$n-1$ equations

Simply set $z_0 = 0$, $z_n = 0$ and solve for the rest
$\Rightarrow$ natural cubic spline

Theorem: Let $f \in C^2[a,b]$ & let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. If S is the natural cubic spline interpolating f at t_i, $i = 0, \to n-1$

then $\int_a^b (S''(x))^2 dx \leq \int_a^b f''(x)^2 dx$

↖ among all f's that fit the data S is the "smoothest"

Proof

Let $g := f - s \iff f = g + s$

$$\Rightarrow \int_a^b (f''(x))^2 dx = \int_a^b (g''(x))^2 dx + \int_a^b (s''(x))^2 dx + 2\int_a^b g''(x) s''(x) dx$$

≥ 0 ↑ ; $? \geq 0$?

If we can show that $\int_a^b g'' s'' dx$ is ≥ 0 we are done

$$\int_a^b g''(x) s''(x) dx = \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} g''(x) s''(x) dx$$

Integration by parts: $(g's'')' = g''s'' + g's''$

$$\Rightarrow \int_{t_{i-1}}^{t_i} g''s'' = \int_{t_{i-1}}^{t_i} (g's'')' - \int_{t_{i-1}}^{t_i} g's''' = (g's'')(t_i) - (g's'')(t_{i-1}) - \int_{t_{i-1}}^{t_i} g's'''$$

$(g's'')(b) - (g's'')(a) = 0$ because $s''(a) = s''(b) = 0$

$$\Rightarrow \sum_{i=0}^{n} \int_{t_i}^{t_i} g''(x)s''(x)\,dx = \sum_{i=0}^{n} (g's')(t_i) - (g's'')(t_{i-1}) - \sum_{i=0}^{n} \int_{t_{i-1}}^{t_i} g's_i'''$$

$$= -\sum_{i=0}^{n} c_i \int_{t_{i-1}}^{t_i} g'(x)\,dx = -\sum_{i=0}^{n} c_i [g(t_i) - g(t_{i-1})]$$

$g(t_i) = f(t_i) - s(t_i) = 0$, $g(t_{i-1}) = 0$

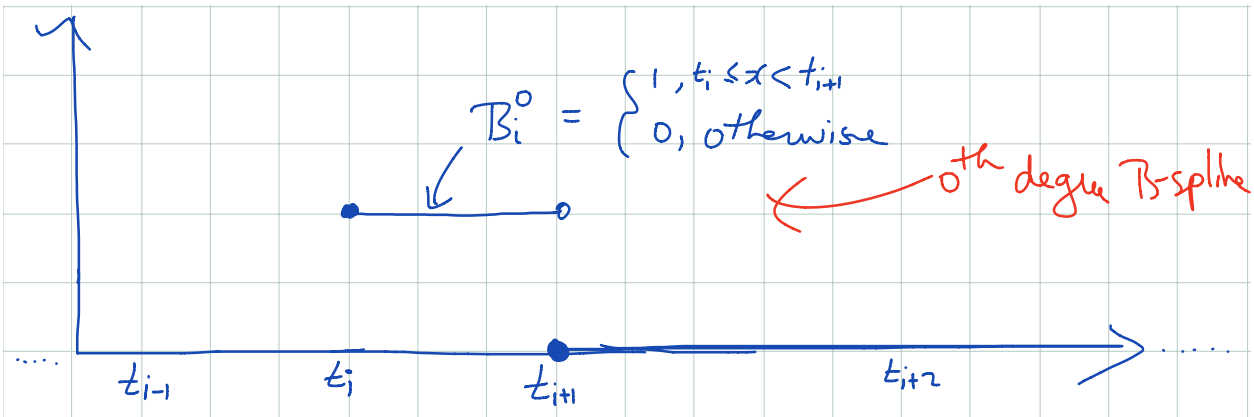
$= 0$!

HW: 6.4: 5, 6, 7, 24

6.5 B-splines:

(Short for basis splines)

- Given the knots, B-splines can be easily calculated (via recurrence relation)



Note:

- $B_i^0(x) \geq 0 \quad \forall i \ \& \ \forall x$
- $\sum_{i \in \mathbb{Z}} B_i^0(x) = 1 \quad \forall x$
- B_i^0 is right continuous
- B_i^0 form a Basis for all splines of degree 0 (for the (same) knots)

Proof: Degree 0 spline $S(x) = c_i$ when $t_i \le x < t_{i+1}$

$$\Rightarrow S(x) = \sum_i c_i B_i^0(x)$$

Higher order B-splines:

$$B_i^k(x) = \underbrace{\left(\frac{x - t_i}{t_{i+k} - t_i}\right)}_{V_i^k(x)} B_i^{k-1}(x) + \underbrace{\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}}_{1 - V_{i+1}^k(x)} B_{i+1}^{k-1}(x)$$

(k: order)

↑ recursively defined.

e.g.: 1st order B-splines:

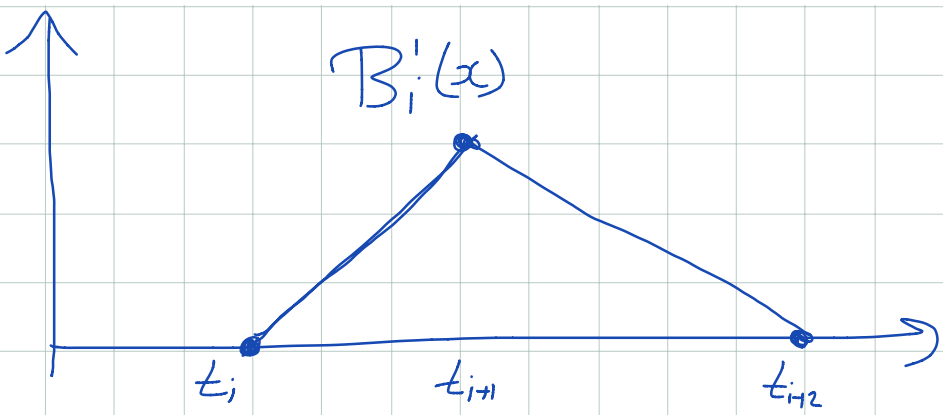
$$B_i^1(x) = V_i^1(x) B_i^0(x) + \left(1 - V_{i+1}^1\right)(x) B_{i+1}^0(x)$$

($V_i^1(x)$: linear, $B_i^0(x)$: 0-degree, $1 - V_{i+1}^1(x)$: linear, $B_{i+1}^0(x)$: 0-degree)

$\Rightarrow B_i^{(1)}(x)$ is a degree 1 polynomial.

$$B_i^1(x) = \frac{x - t_i}{t_{i+1} - t_i} B_i^0(x) + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1}^0(x)$$

$$= \begin{cases} 0 & x < t_i \\ \frac{x - t_i}{t_{i+1} - t_i}, & t_i \le x < t_{i+1} \\ \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}}, & t_{i+1} \le x < t_{i+2} \\ 0, & x \ge t_{i+2} \end{cases}$$



Note: • $B_i'(x)$ supported on (t_i, t_{i+2}) ← 0 otherwise

• $B_i'(x) \geq 0$ ✓

• $B_i'(x)$ is cont's.

• $B_i'(x)$ is differentiable except at t_i, t_{i+1} & t_{i+2}

• $\sum_{i \in \mathbb{Z}} B_i'(x) = 1 \quad \forall x$

↳ **proof:** $\forall x, \exists i : x \in [t_i, t_{i+1})$

$\Rightarrow B_j'(x) = 0 \quad \forall j \notin \{i, i-1\}$

$$\Rightarrow \sum_{j \in \mathbb{Z}} B_j'(x) = B_i'(x) + B_{i-1}'(x)$$

$$= \frac{t_{i+1} - x}{t_{i+1} - t_i} + \frac{x - t_i}{t_{i+1} - t_i} = 1$$

∎

Properties

(I) $B_i^k(x) = 0 \quad \forall x \notin (t_i, t_{i+k+1}) \;, k \geq 1$

(i.e. $B_i^k(x)$ is supported of $[t_i, t_{i+k+1})$

Proof by induction: $B_i^1(x) = 0 \quad \forall x \notin (t_i, t_{i+2})$ (see above Note)

Now, suppose it's true for $k-1$, i.e. $B_i^{k-1}(x) = 0$ $\forall x \notin (t_i, t_{i+k})$

and note that $x \notin (t_i, t_{i+k+1}) \Rightarrow \begin{cases} x \notin (t_i, t_{i+k}) \\ x \notin (t_{i+1}, t_{i+k+1}) \end{cases}$

$\Rightarrow B_i^{k-1}(x) = 0 \;\&\; B_{i+1}^{k-1}(x) = 0$

but

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+k} - t_i}\right) \underbrace{B_i^{k-1}(x)}_{0} + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} \underbrace{B_{i+1}^{k-1}(x)}_{0} = 0.$$

(II) For $k \geq 0$, if $x \in (t_i, t_{i+k+1})$, $B_i^k(x) > 0$

(i.e., $B_i^k(x)$ is +ve on it's support).

Proof: exercise

(III) $\sum_{i=-\infty}^{\infty} c_i B_i^k = \sum_{i=-\infty}^{\infty} \left[c_i V_i^k + c_{i-1}\left(1 - V_i^k\right)\right] B_i^{k-1}$

(recurrence relation of B-splines)

Proof: $\sum_{i=-\infty}^{\infty} c_i B_i^k = \sum_{i=-\infty}^{\infty} c_i \left[V_i^k B_i^{k-1} + \left(1 - V_{i+1}^k\right) B_{i+1}^{k-1}\right]$

$= \sum c_i V_i^k B_i^{k-1} + \sum c_{i-1}\left(1 - V_i^k\right) B_i^{k-1}$ ∎

Great for computation: read in book.

More properties (without proof)

- $\sum_{i=-\infty}^{\infty} B_i^k(x) = 1$
- $\frac{d}{dx} B_i^k(x) = \frac{k}{t_{i+k} - t_i} B_i^{k-1}(x) - \frac{k}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(x)$

  $\forall k \geq 2$ (for $k=1$, it's true except at t_i, t_{i+1}, t_{i+2})
- for $k \geq 1$, $B_i^k \in C^{k-1}(\mathbb{R})$
- $\int_{-\infty}^{x} B_i^k(s)\,ds = \frac{t_{i+k+1} - t_i}{k+1} \sum_{j=1}^{\infty} B_j^{k+1}(x)$

- The set $\{B_{-k}^k, B_{-k+1}^k, \ldots, B_{n-1}^k\}$ is linearly indep. on (t_0, t_n).

HW: 6.5 : 1,2,3,4

6.6 Application of B splines

Let S_n^k denote the family of all splines (i.e. piecewise poly functions) of degree $\leq k$ for the knots $t_0, \ldots, t_n$.

Theorem: the B splines B_i^k with $-k \leq i \leq n-1$ is a basis for S_n^k.

So the dimension for S_n^k is $n+k$

Proof outline

(1) $B_i^k \in S_n^k$ (each B_i^k is a pcwise polynomial for the knot seq.)

(2) $\mathrm{Dim}\left(S_n^k\right) \leq n+k$

we show this by constructing a set of $k+n$ functions that spans S_k^n.

Let $S(x)$ be any function in S_n^k.

on $[t_0, t_1]$, $S(x)$ is a polynomial of degree k. Call it $P_0(x) = \sum_{i=0}^{k} a_i x^i$.

on $[t_1, t_2]$, $S(x)$ is a polynomial $P_1(x)$ of degree k with

$$P_1^{(r)}(t_1) = P_0^{(r)}(t_1) \quad (S \text{ is a spline})$$

for $r = 0, 1, \ldots k-1$

$\Rightarrow (P_1 - P_0)^{(r)}(t_1) = 0$

$(P_1 - P_0)$: Polynomial of degree k $\Rightarrow (P_1 - P_0)(x) = b_1 (x - t_1)^k$

So, on the piece $[t_0, t_1]$, $S(x) = P_0(x)$

on $[t_1, t_2] = S(x) = P_0(x) + (P_1 - P_0)(x)$

So we can write $S(x) = \sum_{i=0}^{k} a_i x^i + b_1 (x-t_1)_+^k$

on $[t_0, t_2]$

$(x-t_1)_+ = \begin{cases} x-t_1, & x \geq t_1 \\ 0, & x < t_1 \end{cases}$

Continuing in this way: $(P_2 - P_1)^{(r)}(t_2) = 0$

$\Rightarrow \cdots$

$$\Rightarrow S(x) = \sum_{i=0}^{k} a_i x^i + \sum_{i=1}^{n-1} b_i (x-t_i)_+^k$$

for $x \in [t_0, t_n]$

So we have $k+1$ a_i's and $n-1$ b_i's

$\Rightarrow \dim(S_n^k) = n+k$.

(3) By the last property of our notes on section 6.5

$\{B_{-k}^k, \ldots, B_{n-1}^k\}$ is linearly indep.

It contains $n+k$ elements and they are all in $S_n^k \Rightarrow$ they form a basis for S_n^k.

Using B-splines for interpolation at points other than the knots.

Suppose we're given (as before) a data set (x_i, y_i), $i = 1, \dots, n$ with $x_1 < x_2 < \cdots < x_n$. We would like to interpolate this data with a spline

$$S(x) = \sum_{j=1}^{n} c_j B_j^k(x)$$

So, we need to find the c_j's (n-unknowns) and we have n data points $\Rightarrow$ n-equations.

$$y_i = \sum_{j=1}^{n} c_j B_j^k(x_i) \iff A\vec{c} = \vec{y}$$

y_i: known; c_j: unknown; $B_j^k(x_i)$: known, call it A_{ij}; A: $n \times n$

This system will have a solution

when A is non-singular.

Theorem: (Schoenberg-Whitney Theorem)

If $x_1 < x_2 < \cdots < x_n$, the matrix A with $A_{ij} = B_j^k(x_i)$ is non-singular if and only if there are no zeros on its diagonal (i.e. $A_{ii} \neq 0 \ \forall i \in \{1, 2, \ldots n\}$)

Example: Consider the knots $t_i = i$ for all $i \in \mathbb{Z}$

Can we use the set of nodes (x_i's) $\{2.7, 3.1, 3.4, 5.1, 6\}$ to interpolate by linear combinations of $B_1^2, B_2^2, B_3^2, B_4^2, B_5^2$?

Solution: By the S-W Theorem, we need to check that the diagonal of the matrix A with $A_{ij} = B_j^2(x_i)$ has no zeros.

That is, we want $B_i^2(x_i) \neq 0$, but this is true provided $i < x_i < i+3$ for $i = 1, 2, \ldots, 5$.
This is indeed the case, so we may use the nodes.

Theorem: Let $\{x_1, \ldots x_{n+k}\} \subset [t_0, t_n]$

with $t_{i-k-1} < x_i < t_i \quad (1 \leq i \leq n+k)$

then arbitrary data can be interpolated by S_n^k.

Proof: The B-splines B_j^k with $-k \leq j \leq n-1$ are a basis for S_n^k on $[t_0, t_n]$.

Recall that $\text{supp}(B_i^k) = (t_i, t_{i+k+1})$

So $x_{i+k+1} \in \text{Supp}(B_i^k)$

& $B_i^k(\underbrace{x_{i+k+1}}_{\text{call this } y_i}) \neq 0 \Rightarrow$ By S-W Theorem we are done

H/W 6.6, #1

Computer Problem: Write a program that takes $n+1$ data points (x_i, y_i) $i = 0, \ldots, n$ and outputs the newton form of the interpolating polynomial.

Review of Interpolation so far:

- Given $n+1$ data points, (x_i, y_i) $i=0,\dots,n$ we can interpolate them with a polynomial of degree $\leq n$.

- If the x_i's are distinct, the polynomial is unique.

  The conventional form: $P_n(x) = \sum_{i=0}^{n} a_i x^i$

  The Newton form: $P_n(x) = \sum_{i=0}^{n} c_i \prod_{j=0}^{i} (x - x_j)$

  can be computed by divided differences

  The Lagrange form: $P_n(x) = \sum_{i=0}^{n} y_i \ell_i(x)$

  each ℓ_i is a polynomial of degree $\leq n$, $\ell_i(x_j) = \delta_{ij}$

- Error in Polynomial Interp:

$$f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

- Hermite Interpolation

  - Given $P^{(j)}(x_i)$ for $0 \leq j \leq k_i - 1$, find P

  - $\exists$ a unique Poly. P of degree $\leq m$ where $m+1 = \sum_{i=0}^{n} k_i$

- Modified Divided Differences.
  to compute newton form.

- Spline Interpolation:

  - Splines of degree k are piecewise polynomials of degree $\leq k$ on $[t_0, t_n]$ with continuous $(k-1)$ derivatives.

  - Natural Cubic Splines: $S''(t_0) = S''(t_n) = 0$

  - Theorem: $\int_{t_0}^{t_n} (S''(x))^2 dx \leq \int_{t_0}^{t_n} (f''(x))^2 dx$

    $\forall f \in C^2[t_0, t_n]$ with $f(t_i) = y_i$

- B-splines:

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+k} - t_i}\right) B_i^{k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(x)$$

$$= V_i^k(x)\, B_i^{k-1}(x) + \left(1 - V_{i+1}^k(x)\right) B_{i+1}^{k-1}(x)$$

- Properties:
  - $B_i^k(x) = 0 \quad \forall x \notin (t_i, t_{i+k+1}), \; k \geq 1$
  - $B_i^k(x) > 0 \quad \forall x \in (t_i, t_{i+k+1}), \; k \geq 0$
  - $\sum_{i=-\infty}^{\infty} c_i B_i^k = \sum_{i=-\infty}^{\infty} \left[c_i V_i^k + c_{i-1}(1 - V_i^k) \right] B_i^{k-1}$
  - $\sum_{-\infty}^{\infty} B_i^k(x) = 1$
  - $B_i^k \in C^{k-1}(\mathbb{R}) \quad \forall k \geq 1$
  - $\{ B_{-k}^k, \dots B_{n-1}^k \}$ is a Basis for S_n^k on $[t_0, t_n]$

- So we can write $S(x) = \sum c_j B_j^k(x)$

$$\underbrace{S(x_i)}_{y_i} = \sum c_j \underbrace{B_j^k(x_i)}_{A_{ij}} \Rightarrow y = Ac$$

need A to be non-singular

S-W Theorem: A nonsingular if & only if $A_{ii} \neq 0 \; \forall i$